Systems of Equations: Consistency and Dependency

A system of equations is **consistent** if it has at least one solution. A system is **inconsistent** if it has no solution.

Two Variables
In a system of two equations in two variables, the equations are **dependent** if one equation is a multiple of the other. **Dependent** systems have an infinite number of solutions – *every* point is a solution.

Three Variables
In a system of three equations in three variables, if the system reduces to two equations in two variables where one equation is a multiple of the other, the system is **dependent**.

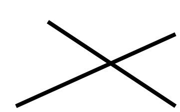
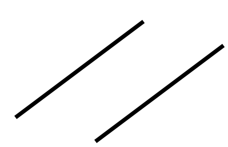
Dependent systems produce the true equation 0 = 0, and are always **consistent**.

If the equations in a system are not **dependent**, then they are **independent**.

If a system is **inconsistent**, solving it produces a false equation, such as 0 = 5.

Systems of equations can be represented (and solved) graphically: equations with two variables, both to the power of 1, would be graphed as straight lines; equations with three variables, all to the power of 1, would be graphed as 3-dimensional planes.

Two Variables

Type of System	Example	Nature of Solutions	Graphic
Dependent, Consistent	$x + y = 2$ $3x + 3y = 6$ **clue** → $0 = 0$	Infinite number of solutions – they are the same line!	One line "on top of" another 
Independent, Consistent	$x + 2y = 5$ $-2x + y = 15$ $x = -5 \quad y = 5$	Unique solution – the lines intersect at one point	Intersection 
Independent, Inconsistent	$2x + 5y = 27$ $6x + 15y = 39$ **clue** → $0 = -42$	No solutions – the lines are parallel	

Evaluating a 2-Variable System using the y = mx + b Forms of the Equations

Put the equations into y = mx + b form and examine the slopes and the intercepts:

1. If the **slopes** are the **same** and the **y-intercepts** are the **same**, then the **lines** are the **same** – the system is dependent and consistent.

$$\begin{cases} x + y = 2 & \rightarrow & & & y = -x + 2 \\ 3x + 3y = 6 & \rightarrow & 3y = -3x + 6 & \rightarrow & y = -x + 2 \end{cases}$$

The slope is -1 and the y-intercept is 2 in both equations – **they are the same line** → the system is **dependent and consistent** (infinite number of solutions).

2. If the **slopes** are the **same** but the **y-intercepts** are **different**, then the **lines** are **parallel** (but not the same line) and **will not intersect** – the system is independent and inconsistent.

$$\begin{cases} 2x + 5y = 27 & \rightarrow & y = -\dfrac{2}{5}x + \dfrac{27}{5} \\ 6x + 15y = 39 & \rightarrow & y = -\dfrac{2}{5}x + \dfrac{39}{15} \end{cases}$$

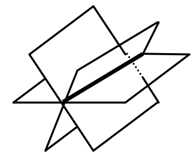
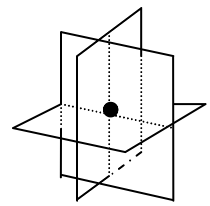
The slope is $-\frac{2}{5}$ in both equations **but** the y-intercepts are different – these lines are parallel (same slope) but different and will not intersect. The system is **independent and inconsistent** (no solution).

3. If the **slopes** are **different**, it doesn't matter what the y-intercepts are – the lines will always intersect at some point. The system is independent and consistent.

$$\begin{cases} x + 2y = 5 & \rightarrow & y = -\dfrac{2}{5}x + \dfrac{5}{2} \\ -2x + y = 15 & \rightarrow & y = 2x + 15 \end{cases}$$

The slopes are different in the two equations – the lines will intersect. The system is **independent and consistent** (one solution).

Three Variables

Type of System	Example	Nature of Solutions	Graphic
Dependent, Consistent	$2x + y + z = 3$ [1] $x - 2y - z = 1$ [2] $3x + 4y + 3z = 5$ [3] [1] + [2] → $3x - y = 4$ [2]*3+[3]→ $\underline{6x - 2y = 8}$ **clue** → $0 = 0$	Infinite number of solutions – the planes intersect along a common line	Intersect along same line 
Independent, Consistent	$4x - 2y - 3z = 5$ $-8x - y + z = -5$ $2x + y + 2z = 5$ $x = 3/2$ $y = -4$ $z = 3$	Unique solution – the planes intersect at one point	Intersect at same point 
Independent, Inconsistent	$y + 3z = 4$ [1] $-x - y + 2z = 0$ [2] $x + 2y + z = 1$ [3] [2] + [3] → $y + 3z = 1$ [4] [1]*(-1)+[4]→ $\underline{-y - 3z = -4}$ **clue** → $0 = -3$	No solution – the planes are either all parallel to each other or they intersect two at a time with no point common to all three